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A CLASS OF REAL COCYCLES HAVING AN ANALYTIC COBOUNDARY MODIFICATION

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ABSTRACT

We define here a certain class of procedures (a.a.c.c.p.) for constructing real valued cocycles over irrational rotations. Each such procedure is realizable over a residual set of possible rotations, and we prove that each such cocycle is cohomologous to a real analytic cocycle. The procedure in Section 3 of [10] is seen to be of this type and hence not only is cohomologous to C^{∞} as is shown there, but is actually cohomologous to a real analytic cocycle. We also show that following the method of [6] a procedure can be given to obtain rank-1 Anzai skew products of mixed spectral type that are real analytic.

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Introduction

In [10] the authors described a construction of two weakly isomorphic but not isomorphic Anzai skew products with C^∞-cocycles. This construction has two features not directly related to the problem of weak isomorphism. First, it is not simply one example that is constructed, but rather a class of examples over a dense G_δ of admissible base rotations. As was discussed there, it is necessary to restrict the rotation number α to be very well approximable by rationals. But moreover it is sufficient, as such very good approximations, that is to say large terms a_k in the continued fraction expansion of $\alpha = [0\colon a_1, a_2, \ldots]$ imply the existence of special Rokhlin towers for T_α whose sets are intervals. The construction was carried out on a series of such towers. More precisely, the ability to carry out the construction depended simply on the existence in the continued fraction expansion of certain $a_n \geq B(a_1, a_2, \ldots, a_{n-1})$ for some rapidly growing function B of the previous terms of the expansion.

Second, for such an α the construction occurred in two steps. In the first step a series of coboundaries φ_k were constructed. The φ_k were step functions, taking on constant values on the levels of a certain Rokhlin tower, and the cocycle $\varphi = \sum_{k=1}^{\infty} \varphi_k$ was shown to have the desired property. In the second step one examines the form of the φ_k and notices that they can be modified by coboundaries to f_k which are smooth cocycles. This remark may seem foolish as of course φ_k is cohomologous to zero. To explain this point we use the language of "fixing sets".

Definition 1: Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system and $f\colon X \longrightarrow \mathbf{R}$ be measurable. We say that f has an "(ε, M)-fixing set" S if $\mu(S) > 1-\varepsilon$ and whenever $x, T^n(x) \in S$, then

$$|f^{(n)}(x)| < M,$$

where $f^{(n)}(x) = f(x) + \cdots + f(T^{n-1}x)$, $n \geq 1$.

THEOREM 1 ([13]): *Assume that T is ergodic. Then a measurable function f is a coboundary if and only if it has an (ε, M)-fixing set for some $0 < \varepsilon < 1$ and $0 < M < \infty$.*

Now in our constructions φ_k is a coboundary which has an $(\varepsilon_k, 0)$-fixing set for some ε_k, but the $\varepsilon_k \longrightarrow 1$, and in fact φ will not be a coboundary. On the other hand, our smooth coboundaries f_k will be such that $\varphi_k - f_k$ has an $(\varepsilon/2^k, M/2^k)$-fixing set S_k for some $0 < \varepsilon < 1$ and $0 < M < \infty$. But then $\sum_{k=1}^{\infty}(\varphi_k - f_k)$

has an (ε, M)-fixing set $(\bigcap_{k=1}^{\infty} S_k)$ and is a coboundary. Our problem reduces to forcing $f = \sum_{k=1}^{\infty} f_k$ to be smooth.

These observations concerning the construction in [10] make it clear that the second part, that of smoothing the cocycle, is only related to certain properties of the construction of the sequence φ_k. We here abstract these properties in the concept of an "almost analytic cocycle construction procedure".

In [10] the cocycle is only smoothed to C^∞. This left the obvious question if one could actually obtain real analytic examples. In the context of $\mathbf{R}^+$-valued cocycles giving rise to weakly mixing special flows, such real-analytic constructions are quite old and known. We will describe such a construction as an a.a.c.c.p. Soon after completing [10] we realized that the necessary computations for the real analytic construction are possible and actually easier than C^∞, and so we work here in this context.

It will be a simple observation that on a perhaps smaller residual set of α's the construction in [10] is an a.a.c.c.p. Hence this class of examples is real analytic. In Section 2 we will modify the method and computations of [6] to give an example of an a.a.c.c.p. that constructs rank-1 automorphism of mixed spectral type.

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1. Notation

We will identify the circle $\mathbf{S^1}$ with $X = [0, 1)$ (mod 1). Therefore, real functions defined on the circle will be identified with periodic-one functions defined on $\mathbf{R}$. Let μ denote Lebesgue measure on X. Assume that $T \colon X \longrightarrow X$ is an irrational rotation, $Tx = x + \alpha$ (mod 1), $x \in X$. Let

$$\alpha = [0; a_1, a_2, \ldots]$$

be the continued fraction expansion of α. The positive integers a_n are said to be the **partial quotients** of α. Put

$$q_0 = 1,\ q_1 = a_1,\ \ q_{n+1} = a_{n+1}q_n + q_{n-1} \quad p_0 = 0,\ p_1 = 1,\ \ p_{n+1} = a_{n+1}p_n + p_{n-1}.$$

The rationals p_n/q_n are called the **convergents** of α and the inequality

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

holds. The following formula

$$q_{n+1}||q_n\alpha|| + q_n||q_{n+1}\alpha|| = 1$$

holds true. Here $||t||$ is the distance of a real number t from the set of integers. By $\{t\}$ we denote the fractional part of t.

Hence, from the continued fraction expansion of α we obtain, for each n, two Rokhlin towers $\xi_n, \overline{\xi}_n$ whose union coincides with the whole circle. For n even

$$\xi_n = \{[0,\{q_n\alpha\}), T[0,\{q_n\alpha\}), \ldots, T^{(a_{n+1}q_n+q_{n-1})-1}[0,\{q_n\alpha\})\},$$

$$\overline{\xi}_n = \{[\{q_{n+1}\alpha\},1), \ldots, T^{q_n-1}[\{q_{n+1}\alpha\},1)\}.$$

Given a subsequence $\{n_k\}$ of natural numbers we will denote

$$I_k = [0,\{a_{2n_k+1}q_{2n_k}\alpha\}), \quad J_t^k = T^{(t-1)q_{2n_k}}(0,\{q_{2n_k}\alpha\}],$$

$t = 1, \ldots, a_{2n_k+1}$. Notice that

$$I_k = \bigcup_{t=1}^{a_{2n_k+1}} J_t^k,$$

and if $l_k = |J_1^k|$ then

$$l_k < \frac{1}{a_{2n_k+1}q_{2n_k}}. \tag{1}$$

We also have

$$||q_{2n_k+1}\alpha|| < \frac{1}{q_{2n_k+1}} < \frac{1}{a_{2n_k+1}q_{2n_k}},$$

so

$$\frac{1-1/a_{2n_k+1}}{q_{2n_k}} \le |I_k| < \frac{1}{q_{2n_k}}.$$

In particular,

$$\text{if } a_{2n_k+1} > 1 \text{ then } \frac{1}{2q_{2n_k}} \le |I_k| < \frac{1}{q_{2n_k}}. \tag{2}$$

Let G be a locally compact abelian metric group with Haar measure m (the only case considered here will be $G = \mathbf{R}$ or $\mathbf{S}^1$ with Lebesgue measure). A measurable function $\overline{\varphi}$: $\mathbf{Z}\times X \longrightarrow G$ is called a **cocycle** if $\overline{\varphi}^{(n+m)}(x) = \overline{\varphi}^{(n)}(x)\cdot\overline{\varphi}^{(m)}(T^n(x))$. Any such is clearly of the form $\overline{\varphi}^{(n)}(x) = \prod_{j=0}^{n-1}\varphi(T^j(x))$, $n \ge 0$, $\overline{\varphi}^{(n)}(x) = (\prod_{j=n}^{-1}\varphi(T^j(x)))^{-1}$, $n < 0$, where $\varphi(x) = \overline{\varphi}(1,x)$ is the "generator" of the cocycle. Abusing language we will refer to φ as "the" cocycle although we will be referring to the cocycle it generates. A cocycle φ determines an automorphism T_φ (called a G-**extension of** T) on $(X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ by

$$T_\varphi(x,g) = (Tx, g\cdot\varphi(x)),$$

($\tilde{\mathcal{B}}$ is the product σ -algebra and $\tilde{\mu} = \mu \times m$). $\mathbf{S}^1$-extensions will be called **Anzai skew products.** A cocycle φ is said to be a **coboundary** (or a G-**coboundary** if we need to emphasize the role of G) if it is of the form

$$\varphi(x) = f(Tx)/f(x)$$

for a measurable function $f\colon X \longrightarrow G$. We say that two cocycles $\varphi, \psi\colon X \longrightarrow G$ are **cohomologous** if φ/ψ is a coboundary.

2. What is an A.A.C.C.P. ("almost analytic cocycle construction procedure") ?

We begin by noticing some simple facts concerning real trigonometric polynomials. Let $Q(x) = \sum_{s=-N}^{N} b_s e^{2\pi i s x}$, where $b_s = \overline{b_{-s}}$, $b_N \neq 0$ be a real trigonometric polynomial. The number N will be called the **degree** of Q. We will denote

$$||Q||_{\mathcal{F}} = \max_{-N\le s\le N} |b_s|, \quad ||Q||_\infty = \sup_{x\in\mathbf{R}} |Q(x)|.$$

Notice that $||Q(\cdot + x_0)||_{\mathcal{F}} = ||Q(\cdot)||_{\mathcal{F}}$ for each $x_0 \in \mathbf{R}$ and $||Q||_\infty$ enjoys the same property.

LEMMA 1: *Let Q_k be real trigonometric polynomials with degrees N_k, $k \geq 1$. If there exists $A > 1$ such that*

$$\sum_{k=1}^{\infty} A^{N_k} ||Q_k||_{\mathcal{F}} < +\infty$$

then the function

$$f(x) = \sum_{k=1}^{\infty} Q_k(x)$$

is real analytic.

Proof: Denote

$$Q_k(x) = \sum_{s=-N_k}^{N_k} b_s^{(k)} e^{2\pi i s x}.$$

Then we formally write

$$f(x) = \sum_{s=-\infty}^{\infty} (\sum_{k=1}^{\infty} b_s^{(k)}) e^{2\pi i s x},$$

where $b_s^{(k)} = 0$ for $|s| > N_k$. We have

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} |b_s^{(k)}| &= \textstyle\sum_{\{k:\ N_k<|s|\}} |b_s^{(k)}| + \sum_{\{k:\ N_k\geq|s|\}} |b_s^{(k)}| \\ &= \textstyle\sum_{\{k:\ N_k\geq|s|\}} |b_s^{(k)}| \leq \sum_{\{k:\ N_k\geq|s|\}} \frac{1}{A^{N_k}} (A^{N_k} ||Q_k||_{\mathcal{F}}) \\ &\leq \textstyle\frac{1}{A^{|s|}} \sum_{k=1}^{\infty} A^{N_k} ||Q_k||_{\mathcal{F}}. \end{aligned}$$

Therefore the Fourier coefficients of f tend to zero exponentially so f is real analytic. ∎

An a.a.c.c.p. is given by a collection of parameters as follows. We are given a sequence $\{M_k\}$ of natural numbers and an array $\{(d_{k,1}, \ldots, d_{k,M_k})\}$, $d_{k,i} \in \mathbf{R}$ satisfying for each k

$$\sum_{i=1}^{M_k} d_{k,i} = 0. \tag{3}$$

Denote $D_k = \max_{1\leq i\leq M_k} |d_{k,i}|$. Choose a sequence $\{\varepsilon_k\}$ of positive real numbers satisfying

$$\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < +\infty, \tag{4}$$

$$\sum_{k=1}^{\infty} \varepsilon_k < 1, \tag{5}$$

$$\varepsilon_k < \frac{1}{D_k^2}, \qquad k = 1, 2, \ldots. \tag{6}$$

Finally, we are given $A > 1$ completing the parameters of the a.a.c.c.p.

We say that this a.a.c.c.p. **is realized over an irrational number** α with continued fraction expansion $[0; a_1, a_2, \ldots]$ if there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that

$$A^{N_k} \frac{D_k M_k ||P_k||_{\mathcal{F}}}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k} \tag{7}$$

and

$$\frac{D_k ||P_k'||_{\infty}}{a_{2n_k+1} q_{2n_k}} < \sqrt{\varepsilon_k}, \tag{8}$$

where $\{P_k\}$ is a sequence of "bump" real trigonometric polynomials, i.e.

$$\left\{\begin{array}{ll}\text{(i)} & \int_0^1 P_k(t)dt = 1,\\ \text{(ii)} & P_k \geq 0,\\ \text{(iii)} & P_k(t) < \varepsilon_k \text{ for each } t \in (\eta_k/2, 1),\end{array}\right. \tag{9}$$

where the η_k's are chosen in such a way that

$$4M_k\eta_k < \frac{\varepsilon_k}{q_{2n_k}} \tag{10}$$

and N_k is the degree of P_k. Finally, $a_{2n_k+1} > 1$ and

$$\frac{1}{a_{2n_k+1}q_{2n_k}} < \frac{1}{2}\eta_k. \tag{11}$$

Using the above parameters define a cocycle

$$\varphi = \sum_{k=1}^{\infty} \varphi_k$$

as follows. In view of (10), (11) (and (1), (2)), in the interval

$$I_k = [0, \{a_{2n_k+1}q_{2n_k}\alpha\})$$

we can choose $w_{k,1}, \dots, w_{k,M_k}$ to be consecutive pairwise disjoint intervals of the same length contained between η_k and $2\eta_k$ such that each $w_{k,i}$ consists of, say, e_k consecutive subintervals J_t^k, where e_k is an odd number. Let $J_{s_{k,i}}^k$ be the central subinterval in $w_{k,i}$ and now define

$$\varphi_k(x) = \left\{\begin{array}{ll} d_{k,i} & \text{if } x \in J_{s_{k,i}}^k,\\ 0 & \text{otherwise.}\end{array}\right.$$

Note that the φ_k's have disjoint supports so φ is well defined.

Let $m_{k,i}$ be determined by $J_{s_{k,i}}^k = T^{m_{k,i}} J_1^k$.

Proposition 1: *The set of α's over which an a.a.c.c.p. is realized is a G_δ and dense subset of the circle.*

Proof: In constructing φ_k we use only information relating parameters

$$[0; a_1, \dots, a_{2n_k}], M_k, D_k, \varepsilon_k.$$

Now, the set $\{M_k, \varepsilon_k, q_{2n_k}\}$ determines an upper bound for our choice of η_k which in turn determines our choice of P_k which finally determines a lower bound for

a_{2n_k+1} to satisfy (7)–(9) and (11). We can then choose an arbitrary n_{k+1} and put arbitrarily $a_{2n_k+2}, \dots, a_{2n_{k+1}}$. Therefore density follows directly from the fact that n_1 can be chosen arbitrarily and G_δ from the fact that for each $k \geq 1$ the set of α's satisfying (7)–(9) and (11) is open. ∎

THEOREM 2: *Suppose that for an irrational α an a.a.c.c.p. is realized. Then there exists an analytic cocycle $f\colon \mathbf{S}^1 \longrightarrow \mathbf{R}$ which is α-cohomologous to φ.*

Proof: Denote

$$\tilde{P}_k(t) = l_k \sum_{r=1}^{M_k} d_{k,r} P_k(t - m_{k,r}\alpha).$$

We are going to prove that the real function

$$f(t) = \sum_{k=1}^{\infty} \tilde{P}_k(t) \text{ is analytic} \tag{12}$$

and is cohomologous to φ. First notice that

$$||\tilde{P}_k||_{\mathcal{F}} \leq l_k D_k M_k ||P_k||_{\mathcal{F}}.$$

Moreover the degree of $\tilde{P}_k$ is not bigger than the degree of P_k. Therefore, by Lemma 1 (with $Q_k = \tilde{P}_k$), (7) and (1) it follows that (12) holds true. To show f and φ cohomologous put

$$S_k = [0,1) \setminus \bigcup_{r=1}^{M_k} \bigcup_{i=0}^{q_{2n_k}-1} T^i w_{k,r}.$$

Now,

$$\mu(S_k) = 1 - \mu\Big(\bigcup_{r=1}^{M_k} \bigcup_{i=0}^{q_{2n_k}-1} T^i w_{k,r}\Big) \geq 1 - 4M_k \eta_k q_{2n_k} \geq 1 - \varepsilon_k$$

in view of (10). So, from Theorem 1, (4) and (5) it follows that it is enough to show that

$$\text{if } x, T^N x \in S_k \text{ then } |(\varphi_k - \tilde{P}_k)^{(N)}(x)| \leq 12 M_k \sqrt{\varepsilon_k}. \tag{13}$$

Denote $g_k = \varphi_k - \tilde{P}_k$, $h_{k,j} = \chi_{J_1^k} T^{-m_{k,j}} - l_k P_k T^{-m_{k,j}}$. We will estimate the sum

$$|g_k(x) + g_k(Tx) + \cdots + g_k(T^{N-1}x)|$$

by $4M_k$ sub-sums of the form

$$\left|\sum_{i=0}^{L_j-1} d_{k,j}h_{k,j}(T^iy)\right|,$$

where

(14) $$0 \le L_j \le a_{2n_k+1}q_{2n_k},$$

(15) $$\begin{cases} \text{neither } y \text{ nor } T^{L_j-1}y \text{ are in the set} \\ \tilde{S}_{k,j} = \bigcup_{i=0}^{q_{2n_k}-1} T^i w_{k,j} \end{cases}$$

and moreover if i_1 and i_2 are the smallest positive integers such that $y \in T^{i_1}J_1^k$ and $T^{L_j-1}y \in T^{i_2}J_1^k$ then

(16) $$i_1 < i_2.$$

Indeed, in our notation

$$g_k = \sum_{j=1}^{M_k} d_{k,j}h_{k,j}.$$

Since (3) holds and $h_{k,j} = \overline{h}_k \circ T^{-m_{k,j}}$, where $\overline{h}_k = \chi_{J_1^k} - l_kP_k$, we have

$$\begin{aligned} g_k^{(N)}(x) &= \sum_{j=1}^{M_k}\sum_{i=0}^{N-1} d_{k,j}\overline{h}_kT^{-m_{k,j}}(x+i\alpha) \\ &= \sum_{j=1}^{M_k}\Bigg(\underbrace{\sum_{i=0}^{m_{k,j}-m_{k,1}-1} d_{k,j}h_{k,j}(x+i\alpha)}_{I_j} \\ &\qquad + \underbrace{\sum_{i=(N-1)-(m_{k,M_k}-m_{k,j})}^{N-1} d_{k,j}h_{k,j}(x+i\alpha)}_{II_j}\Bigg). \end{aligned}$$

Now, $\tilde{S}_{k,j} = T^{m_{k,j}-m_{k,1}}\tilde{S}_{k,1}$ and therefore for the sum I_j we have (14) and (15) with $L_j = m_{k,j} - m_{k,1}$, $y = x$. Similarly, for appropriate L_j and y we have (14) and (15) for the sum II_j, $j = 1, \ldots, M_k$. However, it is not excluded that the set $\{y, \ldots, T^{L_j-1}y\}$ is not entirely contained in ξ_{2n_k}. If this happens then no more than $q_{2n_k-1} + q_{2n_k}$ consecutive points from this set are outside ξ_{2n_k}. Hence by dividing each sum I_j (II_j) into no more than two sums we get that $|g_k^{(N)}(x)|$ can be estimated by the sum of absolute values of no more than $4M_k$ sums each of which satisfies (14), (15) and (16).

Let us take a sum of the form

$$\sum_{i=0}^{N-1} d_{k,j} h_{k,j}(x+i\alpha),$$

where x, $T^{N-1}(x)$ are not in $\tilde{S}_{k,j}$ and (16) is satisfied. We divide this sum into three sums Σ_r, $r = 1,2,3$, where the first sum is from 0 to N_1, where N_1 is the smallest number such that $T^{N_1+1}(x)$ is in $\tilde{S}_{k,j}$, if it exists, the second from N_1+1 to N_2, where N_2 is the biggest (but smaller than N) number such that $T^{N_2}(x)$ is in $\tilde{S}_{k,j}$ and finally the third one from N_2+1 to $N-1$. If N_1 does not exist neither does N_2 and the sum is just Σ_1. Now, since $P_k T^{-m_{k,j}}$ is smaller than ε_k outside $\tilde{S}_{k,j}$,

$$\begin{aligned}
|\Sigma_1 + \Sigma_3| \leq & |\Sigma_1| + |\Sigma_3| \\
= & \Big|\sum_{i=0}^{N_1} d_{k,j}\chi_{J_1^k} T^{-m_{j,k}}(x+i\alpha) - \sum_{i=0}^{N_1} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| \\
+ & \Big|\sum_{i=N_2+1}^{N-1} d_{k,j}\chi_{J_1^k} T^{-m_{j,k}}(x+i\alpha) - \sum_{i=N_2+1}^{N-1} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| \\
= & \Big|\sum_{i=0}^{N_1} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| + \Big|\sum_{i=N_2+1}^{N-1} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| \\
\leq & D_k l_k a_{2n_k+1} q_{2n_k} \varepsilon_k < D_k \varepsilon_k < \sqrt{\varepsilon_k} \quad \text{by (6).}
\end{aligned}$$

Finally

$$\begin{aligned}
|\Sigma_2| = & \Big|\sum_{i=N_1+1}^{N_2} d_{k,j}\chi_{J_1^k} T^{-m_{j,k}}(x+i\alpha) - \sum_{i=N_1+1}^{N_2} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| \\
= & \Big|d_{k,j}\cdot 1 - \sum_{i=N_1+1}^{N_2} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha)\Big| \\
= & \Big|\sum_{i=N_1+1}^{N_2} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha) - d_{k,j}\int_0^1 P_k(t)\,dt\Big| \\
\leq & \Big|\sum_{i=N_1+1}^{N_2} d_{k,j} l_k P_k T^{-m_{j,k}}(x+i\alpha) - \int_{\tilde{S}_{k,j}} P_k(t)\,dt\Big| + \Big|\int_{[0,1)\setminus \tilde{S}_{j,k}} P_k(t)\,dt\Big| \\
\leq & D_k \|P_k'\|_\infty l_k + D_k \varepsilon_k < 2\sqrt{\varepsilon_k}
\end{aligned}$$

in view of (6) and (8). We conclude that (13) holds true, so the proof is complete.
∎

3. Comments and applications

1. First notice that the method in Section 3 of [10] for constructing Anzai skew products that are weakly isomorphic but not isomorphic are realized as an a.a.c.c.p. Therefore not only a C^∞-construction as in [10] but an analytic example of two diffeomorphisms (preserving Lebesgue measure on the two dimensional torus) which are weakly isomorphic but not isomorphic can be carried out. Notice also that in [10], a C^∞-coboundary modification was possible only for an infinite sums of step cocycles which were uniformly bounded. Our new method omits this difficulty and can be applied to unbounded cocycles.

By a simple trick on the 4-dimensional torus we can obtain positive entropy analytic examples of weakly isomorphic transformations that are not isomorphic. Indeed, let T_1, T_2 be two zero entropy analytic diffeomorphisms on the 2-torus that are weakly isomorphic but not isomorphic. Let S be an ergodic continuous group automorphism on the 2-torus. Then measure-theoretically S is Bernoulli. Obviously, $T_1 \times S$ and $T_2 \times S$ are weakly isomorphic. They cannot be isomorphic however since T_i is the Pinsker algebra of $T_i \times S$, $i = 1,2$. This example would be much more interesting if the systems were K. Our method seems incapable of giving even mixing examples.

We recall that on the 2-dimensional torus all positive entropy smooth automorphisms that are weakly isomorphic are in fact isomorphic.

2. Suppose that for certain parameters $\{M_k\}, \{\varepsilon_k\}, A$ and $\{(d_{k,1}, \ldots, d_{k,M_k})\}$, $\sum_{i=1}^{M_k} d_{k,i} = 0$ we have an irrational number α over which this a.a.c.c.p. is realized. This means that the cocycle φ is cohomologous to f which is analytic. Choose an integer p such that if we denote $\Phi = \varphi + p$, $F = f + p$ then $F > 0$. Consider now the special flow over the irrational rotation by α with the roof function F. Such a flow is weakly mixing iff for no real number $r \neq 0$ can we solve the functional equation

$$\exp(2\pi i r F(x)) = c_r(Tx)/c_r(x), \tag{17}$$

where $c_r\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is measurable ([7]). Of course, if in (17) we replace F by Φ we obtain exactly the same functional equations to consider.

Proposition 2: *There exists an a.a.c.c.p. and an α over which it is realized so that the special flow over rotation by α with the roof function F is weakly mixing.*

Hence there exists a special flow over an irrational rotation with an analytic roof function which is weakly mixing.

In order to prove this proposition we will need a lemma whose proof appeared in [10] (see Lemma 3 in [10]).

LEMMA 2: *For an arbitrary a.a.c.c.p. and α over which it is realized and for any k, the cocycle φ is constant on each interval $T^i(I_k)$, $i = 1, \ldots, q_{2n_k} - 1$. Moreover, if we put $b_{k,i} = \varphi \mid T^i(I_k)$ then*

$$\sum_{i=1}^{q_{2n_k}-1} b_{k,i} = 0.$$

Proof of Proposition 2: According to the definition of an a.a.c.c.p.

$$\varphi \mid J_i^k = e_{k,i}, \quad i = 1, \ldots, a_{2n_k+1},$$

where for $i = s_{k,j}$, $j = 1, \ldots, M_k$, we have $e_{k,i} = d_{k,j}$ and zero for the remaining values of i.

In view of the above remarks all we need to show is that the functional equation

$$\exp(2\pi i r \varphi(x)) = \exp(-2\pi i p r) c_r(Tx)/c_r(x) \tag{18}$$

has no measurable solution c_r: $\mathbf{S}^1 \longrightarrow \mathbf{S}^1$ for any $r \in \mathbf{R} \setminus \{0\}$ if the sequence of parameters $\{(d_{k,1}, \ldots, d_{k,M_k})\}$ satisfies some additional properties. Below, we will indicate one of a variety of such possibilities.

Select the intervals $w_{k,1}, \ldots, w_{k,M_k}$ to be pairwise disjoint and moreover so that there exists $\theta > 0$ with

$$s_{k,i+1} - s_{k,i} \geq \frac{\theta}{M_k} a_{2n_k+1}, \quad i = 0, \ldots, M_k, \tag{19}$$

where $s_{k,0} = 1$, $s_{k,M_k+1} = a_{2n_k+1}$. Partition $\mathbf{N}$ into

$$\mathbf{N} = \bigcup_{t=1}^{\infty} \mathbf{N}_t \text{ where each } \mathbf{N}_t \text{ is infinite.} \tag{20}$$

Let $\mathbf{Q} = \{\beta_t\colon t = 1, 2, \ldots\}$ be the set of all rationals. Select $\{(d_{k,1}, \ldots, d_{k,M_k})\}$ so that there exists an $\eta > 0$ with

$$\begin{cases} |e^{2\pi i \beta_t d_{k,j}} - e^{2\pi i \beta_t d_{k,j+1}}| \geq \eta \\ \text{for each } k \in \mathbf{N}_t \text{ and } j = 1, \ldots, M_k - 1 \text{ for all } t. \end{cases} \tag{21}$$

Suppose now that there exists $r \neq 0$ such that (18) can be solved. Take a positive number $\delta < \eta/16$. In view of the measurability of c_r, there exists k_0 such that for all $k \geq k_0$, there exist at least $(1-\delta)q_{2n_k}$ "good" intervals of the form $T^i I_k$, $i = 1, \ldots, q_{2n_k} - 1$ on which the values of function c_r are contained in a ball of radius δ except for a subset of measure $\frac{\delta}{|I_k|}$. Hence, there must exist an i_0 such that for at least $(1-2\delta)a_{2n_k+1}$ "good" intervals of ξ_k contained in $T^{i_0}(I_k)$ the values of function c_r are within 2δ from a *common* value, say a, except for $\delta/|J_1^k|$ of the mass of each such interval. Let $A = T^{i_0} J^k_{r_0}$ denote a "good" interval. Consider an arbitrary "good" interval which is on the right of A. It is of the form $T^{mq_{2n_k}} A$, for some m.

From the above and (18), we can find an $x \in A$ satisfying the following:

$$|c_r(x) - a| < 2\delta,$$

$$|c_r(T^{mq_{2n_k}} x) - a| < 2\delta$$

and

$$(e^{2\pi i r\varphi})^{(mq_{2n_k})}(x) = e^{-2\pi i r p m q_{2n_k}} \frac{c_r(T^{mq_{2n_k}} x)}{c_r(x)}$$

(actually the set of such points have positive measure). Consequently

$$|(e^{2\pi i r\varphi})^{(mq_{2n_k})}(x) - e^{-2\pi i r p m q_{2n_k}}| < 4\delta.$$

Applying Lemma 1, we obtain that

$$|e^{2\pi i r(e_{k,r_0+1}+\cdots+e_{k,r_0+m})} - e^{-2\pi i r p m q_{2n_k}}| < 4\delta. \tag{22}$$

Now, since the number of "good" subintervals contained in $T^{i_0}(I_k)$ is at least $(1-2\delta)a_{2n_k+1}$ and (19) holds, for sufficiently small δ there exist i_1, j_1 and m such that

$$\begin{cases} s_{k,j_1-1} < i_1 < s_{k,j_1} \quad & s_{k,j_1} < i_1 + m < s_{k,j_1+1} \\ s_{k,j_1} < i_2 < s_{k,j_1+1} \quad & s_{k,j_1+1} < i_2 + m < s_{k,j_1+2} \\ T^{i_0} J^k_{i_1}, T^{i_0} J^k_{i_1+m}, T^{i_0} J^k_{i_2}, T^{i_0} J^k_{i_2+m} \text{ are "good" intervals.} \end{cases} \tag{23}$$

From our definition of a.a.c.c.p., (23) and (22) it follows that

$$|e^{2\pi i r d_{k,j_1}} - e^{2\pi i r d_{k,j_1+1}}| < 8\delta. \tag{24}$$

Now, choose t such that $|e^{2\pi i\beta_t x} - e^{2\pi i r x}| < \eta/4$ for all $x \in [0,1)$. Then, in view of (24) we obtain that

$$|e^{2\pi i\beta_t d_{k,j_1}} - e^{2\pi i\beta_t d_{k,j_1+1}}| < \eta/2 + 8\delta.$$

By considering only $k \in \mathbf{N}_t$ we get a contradiction to (21). ∎

Remark 1: For other constructions of weakly mixing special flows over irrational rotations with analytic roof functions see [7], [3] chapter 16. In subsection 3 we will show that Katok's method [7] cannot lead to our examples.

3. M. Herman [5] has shown that if $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ is an analytic function periodic of period 1 which is not a trigonometric polynomial then there exists a residual set of irrational numbers such that for each α from this set the diffeomorphism

$$(e^{2\pi ix}, e^{2\pi iy}) \mapsto (e^{2\pi i(x+\alpha)}, e^{2\pi iy}e^{2\pi if(x)}) \tag{25}$$

has partially continuous spectrum, more precisely he has shown that the cocycle $e^{2\pi if}$ is not cohomologous to any constant. Here, using ideas from [7] we prove the following stronger result.

THEOREM 3: *Suppose that $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ is a $C^{1+\delta}$, $\delta > 0$ periodic function of period 1 which is not a trigonometric polynomial. Then there exists a residual set $\mathcal{A}$ of irrational numbers such that for each $\alpha \in \mathcal{A}$ the diffeomorphism given by (25) has partially continuous spectrum and is coalescent.*

(We recall that an automorphism is said to be *coalescent* if each measure-preserving transformation commuting with it is invertible.) Since a coalescent automorphism cannot have a weakly isomorphic factor which is not isomorphic to it, this proves that the construction of weakly isomorphic diffeomorphisms of the form (25) cannot be achieved over a residual set of irrational rotations by using a fixed analytic function (and, as is well-known, cannot be achieved over a set of full measure) and shows how the constructions using a.a.c.c.p.'s are distinct from those obtained by Theorem 3.

In order to prove Theorem 3 we will need some auxiliary results.

LEMMA 3: *Given an infinite set $\{q_n\}$ of natural numbers and a positive real valued function $r = r(q_n)$ the set*

$$\mathcal{A} = \Big\{\alpha \in [0,1)\colon \text{ for infinitely many } n \quad \text{we have } \Big|\alpha - \frac{p_n}{q_n}\Big| < r(q_n), \\ \text{where } p_n/q_n \text{ are convergents of } \alpha\Big\}$$

is residual.

Proof: Set $S(q) = \{1 \le j < q\colon \gcd(j,q) = 1\}$ and denote

$$\mathcal{A}_N = \bigcup_{n=N}^{\infty} \bigcup_{j \in S(q_n)} \Big(\frac{j}{q_n} - \tilde{r}(q_n), \frac{j}{q_n} + \tilde{r}(q_n)\Big),$$

where $\tilde{r}(q_n) = \min(r(q_n), 1/2q_n^2)$. Obviously $\mathcal{A}_N$ is open. We will show that it is also dense.

For $n \ge 1$ let $S(n) = \{t_1 < t_2 < \cdots < t_{\phi(n)}\}$. Denote

$$l_n = \max(|t_{j+1} - t_j|\colon j = 1, \dots, \phi(n) - 1).$$

Then we have

$$\lim_{n\to\infty} \frac{l_n}{n} = 0. \tag{26}$$

Indeed, let $n = p_1^{s_1} \cdot \ldots \cdot p_k^{s_k}$, where p_i are primes and $s_i > 0, i = 1, \dots, k$. Put also $p_0 = 1$. Denote $m_n = p_0 p_1 \cdots p_{k-1}$. Then $\lim_{n\to\infty}(m_n/n) = 0$. Now, consider the set $\mathcal{N} = \{1, m_n+1, 2m_n+1, \dots, rm_n+1\}$, where $r = n/m_n - 1$. Note that p_k can divide only one of the numbers $im_n + 1, (i+1)m_n + 1$ and moreover no number from $p_1, \dots, p_{k-1}$ can divide any number from $\mathcal{N}$. Consequently, in $\mathcal{N}$ for each pair $im_n + 1, (i+1)m_n + 1$ one number is coprime with n. Since $l_n \le 2m_n$, (26) follows.

Now, $\mathcal{A} = \bigcap_{N=1}^{\infty} \mathcal{A}_N$ and the latter set is residual. To complete the proof it is enough to apply the Legendre theorem: if $\gcd(a,b) = 1$ and $|\alpha - \frac{a}{b}| < 1/2b^2$ then $\frac{a}{b}$ is a convergent of α. ∎

Let $\{a_n\}$ be a summable sequence of nonnegative numbers such that $a_n > 0$ for infinitely many n. Denote

$$\varepsilon_n = a_n/(a_n + a_{2n} + a_{3n} + \cdots).$$

Note that ε_n may go to zero (for example $a_n = 1/n\log^2 n$). In the proof of Theorem 3 we must consider sequences for which ε_n does not tend to zero, for example $a_n = \mathrm{o}(1/n^{1+\delta})$ as the following lemma shows.

LEMMA 4: *If $a_n = \mathrm{o}(g(n))$, where $g(kn) \le g(k)g(n)$ and $\sum_{k=1}^{\infty} g(k) = C < \infty$ then $\{\varepsilon_n\}$ does not go to zero.*

Proof: Choose $n_1 \ge 1, \delta_1 > 0$ so that

$$\frac{a_{n_1}}{g(n_1)} > \delta_1 \text{ and } \frac{a_n}{g(n)} \le \delta_1 \text{ for all } n > n_1. \tag{27}$$

Hence $\sum_{k=2}^{\infty} a_{kn_1} \leq \sum_{k=2}^{\infty} g(kn_1)\delta_1 \leq C\delta_1 g(n_1) < Ca_{n_1}$. Then choose $n_2 > n_1$ and $\delta_2 > 0$ to have (27) with n_2 instead of n_1. This can be repeated infinitely many times and for the chosen subsequence $\{n_k\}$ we have $\varepsilon_{n_k} \geq 1/(1+C)$. ∎

Assume, now, that $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ (periodic of period 1) is in $L^2(\mathbf{S}^1)$. Denote by

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x}$$

the Fourier expansion of f. We have then $f_{-n} = \overline{f_n}$. Now, if $f \in C^{1+\delta}, \delta > 0$ then $f_n = \mathrm{o}(1/n^{1+\delta'})$ for each $0 < \delta' < \delta$ so directly from Lemma 4 we obtain the following.

Corollary 1: *Suppose that f is in $C^{1+\delta}$, $\delta > 0$ and is not a trigonometric polynomial. Then there exist a constant $c > 0$ and an infinite increasing sequence $\{q_n\}$ of natural numbers such that for each n*

$$|f_{q_n}| > c \sum_{m=1}^{\infty} |f_{mq_n}|.$$

In order to conclude we will need the following small extension of a theorem of Katok. We include a proof as the preprint of Katok is not readily available.

Theorem 4 (A. Katok, [7]): *Let $f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x}$ be a $C^{1+\delta}(\mathbf{S}^1), \delta > 0$ function with zero mean. Denote by T an irrational rotation by $e^{2\pi i \alpha}$. Assume that for a sequence $\{p_n/q_n\}$ of rational numbers we have*

$$\frac{|f_{q_n}|}{\sum_{k=1}^{\infty} |f_{kq_n}|} > c > 0 \tag{28}$$

and

$$\frac{\left|\alpha - \dfrac{p_n}{q_n}\right| q_n}{|f_{q_n}|} \longrightarrow 0. \tag{29}$$

Then for each $\lambda \in \mathbf{S}^1$ the cocycle $\lambda e^{2\pi i f(\cdot)}$ is not a T-coboundary.

Proof: We begin the proof with certain general remarks. Let $(X, \mathcal{B}, \mu)$ be a probability space.

LEMMA 5: *There exists an increasing function $\xi_1(\kappa) = 3\kappa^2/(4-\kappa^2)$, $0 \le \kappa \le 1$ such that if $g \in L^\infty(X,\mu)$, $0 < M = ||g||_2 \le ||g||_\infty \le P$ then*

$$\mu\{x \in X\colon |g(x)| \ge M/2\} \ge \xi_1(M/P).$$

Proof: Denote $p = \mu\{x \in X\colon |g(x)| \ge M/2\}$. We have

$$\begin{aligned} M &= \Big(\int_{\{x\colon |g(x)| \ge M/2\}} |g(x)|^2 d\mu(x) + \int_{\{x\colon |g(x)| < M/2\}} |g(x)|^2 d\mu(x) \Big)^{1/2} \\ &\le \Big(P^2 p + \frac{M^2}{4}(1-p) \Big)^{1/2} \end{aligned}$$

Therefore

$$p \ge \frac{3(M/P)^2}{4-(M/P)^2}$$

and the lemma follows. ∎

As a consequence we have the following

LEMMA 6: *There exists increasing positive $\xi_2(\kappa)$, $0 \le \kappa \le 1$ with the following property. Suppose that $\{g_n\} \subset L^\infty(X,\mu)$ with $0 < M_n = ||g_n||_2 \le ||g_n||_\infty \le P_n$ and $\int_X g_n d\mu = 0$. Assume that there exists $\kappa > 0$ such that for each n*

$$\frac{M_n}{P_n} \ge \kappa.$$

Then

$$\mu\{x\colon g_n(x) \ge \xi_1(\kappa) M_n/2\} \ge \xi_2(\kappa)$$

and

$$\mu\{x\colon g_n(x) \le -\xi_1(\kappa) M_n/2\} \ge \xi_2(\kappa).$$

Proof: We already know from Lemma 5 that

$$\mu\{x\colon |g_n(x)| \ge M_n/2\} \ge \xi_1(\kappa),$$

so for a first case, suppose

$$\mu\{x\colon g_n(x) \ge M_n/2\} \ge \xi_1(\kappa)/2$$

and moreover ($\xi_1(\kappa) \le 1$)

$$\mu\{x\colon g_n(x) \ge \xi_1(\kappa) M_n/2\} \ge \xi_1(\kappa)/2.$$

Now, if we denote $q = \mu\{x\colon g_n(x) \le -\xi_1(\kappa)M_n/2\}$ then from the fact that g_n has zero mean we obtain

$$\frac{M_n}{2}\frac{\xi_1(\kappa)}{2} \le P_n q + \xi_1(\kappa)\frac{M_n}{2}\Big(1 - \frac{\xi_1(\kappa)}{2} - q\Big).$$

Hence

$$q \ge \frac{\kappa(\xi_1(\kappa))^2/4}{1 - \kappa\xi_1(\kappa)/2},$$

which completes the proof of the lemma, the other case being precisely symmetric. ∎

We now complete the proof of Theorem 4. We have

$$f^{(q_n)}(x) = \sum_{m=-\infty}^{\infty} f_m \frac{1 - e^{2\pi i m q_n \alpha}}{1 - e^{2\pi m\alpha}} e^{2\pi i m x}.$$

Put $\alpha_n = p_n/q_n$. If, for $k > 0$ we denote

$$f_1^{(k)}(x) = f(x) + f(x + \alpha_n) + \cdots + f(x + (k-1)\alpha_n)$$

then

$$f_1^{(q_n)}(x) = q_n \sum_{l=-\infty}^{\infty} f_{lq_n} e^{2\pi i l q_n x}.$$

Denote

$$M_n = \|f_1^{(q_n)}\|_2, \qquad P_n = q_n \sum_{l=-\infty}^{\infty} |f_{lq_n}| \ge \|f_1^{(q_n)}\|_\infty, \qquad d_n = |\alpha - p_n/q_n|.$$

Obviously, $M_n \le P_n$. Since $q_n|f_{q_n}| < M_n$, by (28) it follows that

$$P_n \le c^{-1}M_n. \tag{30}$$

Thus any ratio of M_n, P_n, and $q_n|f_{q_n}|$ is bounded away from zero independant of n.

Since $f \in C^1$,

$$\Big|\sum_{k=0}^{q_n-1} f(x + k\alpha) - f(x + k\alpha_n)\Big| = \Big|\sum_{k=0}^{q_n-1} f'(\xi_k)(k\alpha - k\alpha_n)\Big| \le c' d_n q_n^2,$$

where c' is a constant depending on f but not on n. Now, in view of (28), (29) and (30) it follows that the L^2-norm and the uniform norm of $f^{(q_n)}$ are of the same order. This function has zero mean. Therefore in view of Lemma 6 it has to have both positive and negative values of that order on sets of measure bounded away from zero by a constant independent of n. Up to an error of order $d_n q_n^2$ the function $f^{(q_n)}$ coincides with the function $f_1^{(q_n)}$.

Let us now take $K_n = [\varepsilon_0/M_n]$, where ε_0 is a sufficiently small constant. If $M_n \to 0$ this will grow with order $\varepsilon_0/q_n|f_{q_n}$ and otherwise is bounded. We have

$$f^{(K_n q_n)}(x) = \sum_{j=0}^{K_n-1} \sum_{k=0}^{q_n-1} f((x + jq_n\alpha) + k\alpha) = \sum_{j=0}^{K_n-1} f^{(q_n)}(x + jq_n\alpha).$$

Replacing in the last expression $f^{(q_n)}$ by $f_1^{(q_n)}$ we allow an error of order $d_n q_n^2 K_n$ $= \mathrm{o}(1)$. On the other hand, we have

$$\sum_{j=0}^{K_n-1} f_1^{(q_n)}(x + jq_n\alpha) = q_n \sum_{l=-\infty}^{\infty} f_{lq_n} \sum_{j=0}^{K_n-1} e^{2\pi ijlq_n^2\alpha} e^{2\pi ilq_n x}. \tag{31}$$

We want again to compare the uniform and L^2-norm, this time for $f_1^{(K_n q_n)}$. The uniform norm does not exceed $K_n P_n$. The L^2-norm is greater than

$$q_n \left| f_{q_n} \sum_{j=0}^{K_n-1} e^{2\pi ijq_n^2\alpha} \right|.$$

By (28) and (30), for $0 \le j \le K_n - 1$

$$|e^{2\pi ijq_n^2\alpha} - 1| \le jq_n^2|\alpha - \alpha_n| \le K_n q_n^2 d_n = \mathrm{o}(1).$$

Hence if n is large enough,

$$\left| \sum_{j=0}^{K_n-1} e^{2\pi ijq_n^2\alpha} \right| \ge K_n/2.$$

Therefore by (29) and the definition of K_n the L^2-norm of $f_1^{(K_n q_n)}$ is greater than a certain constant which depends on f and ε_0 and can be arbitrarily small by a choice of ε_0. What is important is that the ratio of the upper estimate of the uniform norm and the lower estimate for the L^2-norm is a constant independent of q_n and ε_0. Since the mean of (31) is zero, by Lemma 6 it reaches both positive and negative values of order ε_0 on sets of measure separated from zero. Taking into account the remark about the error we conclude that the same is true for

$f^{(K_n q_n)}$. But by (29) and our choice of K_n, the sequence $\{K_n q_n\}$ is a rigidity time for T. Therefore if the cocycle $e^{2\pi i f}$ is cohomologous to a constant then

$$\left|\int_0^1 e^{2\pi i f^{(K_n q_n)}}(x)dx\right| \longrightarrow 1$$

for a subsequence of $\{K_n q_n\}$. For a suitable choice of ε_0 we obtain a contradiction. ∎

Proof of Theorem 3: Without loss of generality we may assume $\int f = 0$. According to Corollary 1 we can choose a subsequence denoted by $\{q_n\}$ such that for a certain constant $c > 0$ and each n

$$|f_{q_n}| > c \sum_{m=1}^{\infty} |f_{mq_n}|. \tag{32}$$

Now, applying Lemma 3 with $r(q_n) = o(|f_{q_n}|/q_n)$, there exists a residual set $\mathcal{A} \subset [0,1)$ of irrational numbers such that for each $\alpha \in \mathcal{A}$ there exists an infinite subsequence of $\{q_n\}$ satisfying (29) in Theorem 3. Obviously, in view of (32) for this subsequence (28) will be satisfied as well.

Let $Tx = x + \alpha$ be an irrational rotation, where $\alpha \in \mathcal{A}$. Recall that if $T_{e^{2\pi i f}}$ is not coalescent then for some $k \in \mathbf{Z} \setminus \{-1, 0, 1\}$ and for some $\beta \in [0,1)$ the cocycle $e^{2\pi i F}$, where

$$F(x) = f(x+\beta) - kf(x)$$

is an $\mathbf{S^1}$-coboundary (see e.g. [10]). However $F(x) = \sum_{n\neq 0} F_n e^{2\pi i n x}$, where

$$F_n = (e^{2\pi i n\beta} - k) f_n$$

and since $|k| \geq 2$, the cocycle F will satisfy the assumptions of Theorem 4 with the same sequence $\{p_n/q_n\}$ as for f and a possibly smaller c, hence $e^{2\pi i F}$ is not a coboundary. ∎

4. All the special flows constructed in 2. must be rigid in the sense of the following definition.

Definition 2: A flow $\{S^t\}$ is said to be **rigid** if there exists a sequence $t_n \longrightarrow \infty$ such that $S^{t_n} \longrightarrow Id$ weakly.

Obviously, if a flow is rigid it is not mixing. Let $Tx = x + \alpha$ be an irrational rotation. Let $f\colon \mathbf{S^1} \longrightarrow \mathbf{R}$ be a positive integrable function. Denote by $\{S^t\} = T^f$

the special flow over T with the roof function f. In [9], it is shown that if f has bounded variation then the corresponding special flow is not mixing. More is true for the flows we construct since recently in [11] it has been shown that if the Fourier coefficients of f are $o(1/n)$ then the corresponding special flow is rigid.

5. We will construct an a.a.c.c.p. giving rise to an analytic rank-1 Anzai skew product with mixed spectrum. We will make use of some ideas from [6].

Suppose that we are given sequences $\{b_k\}, \{c_k\}, \{u_k\}, \{s_k\}$ of natural numbers and that for an a.a.c.c.p. the following conditions are satisfied:

$$s_k \mid a_{2n_k+1}, \quad u_k \mid s_k, \quad u_k = 2v_k, \quad k \geq 1.$$

Assume that $w_{k,j} \subset \tilde{w}_{k,j} = J^k_{(j-1)v_k+1} \cup \cdots \cup J^k_{jv_k+1}$, $j = 1, \cdots, a_{2n_k+1}/v_k$. Hence $M_k = a_{2n_k+1}/v_k$. The only nonzero values $d_{k,j}$ of φ_k will be defined on the center interval $J^k_{s_{k,j}}$ in $\tilde{w}_{k,j}$. More precisely we put

$$d_{k,2q\frac{s_k}{u_k}-1} = \frac{b_k u_k}{s_k} - b_k,$$

$$d_{k,2q\frac{s_k}{u_k}} = \frac{c_k u_k}{s_k} - c_k$$

for $q = 1, \ldots, a_{2n_k+1}/s_k$ and for the remaining values of j

$$d_{k,j} = \frac{b_k u_k}{s_k} \text{ for } j \text{ odd} \quad \text{and}$$

$$d_{k,j} = \frac{c_k u_k}{s_k} \text{ for } j \text{ even.}$$

Supposing $\mathbf{N}_t$ satisfies (20) we require that the sequences $\{b_k\}, \{c_k\}, \{u_k\}, \{s_k\}$ satisfy the following condition: there exists an $\eta > 0$ such that for each integer t the set

$$\mathbf{N}_t = \{k: |e^{2\pi i t\frac{c_k u_k}{s_k}} - e^{2\pi i t\frac{b_k u_k}{s_k}}| \geq \eta\} \text{ is infinite.} \tag{33}$$

Suppose now that for an a.a.c.c.p. satisfying all the above conditions the Anzai skew product $T_{e^{2\pi i\varphi}}$ has an eigenvalue λ which is not of the form $e^{2\pi i m\alpha}, m \in \mathbf{Z}$. Then (see for instance [2]) there exists a nonzero integer t such that the functional equation

$$e^{2\pi i t\varphi(x)} = \lambda\frac{h(Tx)}{h(x)}$$

has a measurable solution $h: \mathbf{S}^1 \longrightarrow \mathbf{S}^1$. By applying arguments from the proof that special flows in 2. were weakly mixing (notice that (19) is satisfied for our a.a.c.c.p.) we obtain a contradiction.

PROPOSITION 3: *There exists an a.a.c.c.p. such that the corresponding Anzai skew product has rank-1 and partly continuous spectrum.*

Proof: We will put some more restrictions on our a.a.c.c.p. to get $T_{e^{2\pi i\varphi}}$ has rank-1. According to the definition of rank-1 (see [12]) it is enough to find a sequence $\mathcal{R}_k = \{F_k, \dots, (T_{e^{2\pi i\varphi}})^{h_k-1}F_k\}$ tending to the partition into points. We recall also that rank-1 implies ergodicity.

We will additionally assume that

$$\frac{1}{a_{2n_{k+1}+1}} = \mathrm{o}\Big(\frac{1}{u_k q_{2n_k}}\Big), \tag{34}$$

$$l_k s_k = \mathrm{o}\Big(\frac{1}{u_k q_{2n_k}}\Big), \tag{35}$$

$$\begin{cases} \frac{(b_k+c_k)u_k}{s_k} = \frac{\tilde{p}_k}{\tilde{q}_k}, \text{ where} \\ \tilde{p}_k < \tilde{q}_k,\ \gcd(\tilde{p}_k, \tilde{q}_k) = 1 \text{ and} \\ \tilde{q}_k \longrightarrow \infty. \end{cases} \tag{36}$$

(Notice that (35) will hold if we replace s_k by $\tilde{q}_k$.) Define

$$\tilde{B}_k = J_1^k \cup J_{u_k+1}^k \cup \dots \cup J^k_{\frac{a_{2n_k+1}}{u_k}+1}.$$

Notice that the sets $\tilde{B}_k, T\tilde{B}_k, \dots, T^{u_k q_{2n_k}-1}\tilde{B}_k$ are pairwise disjoint, so we obtain a sequence

$$\varrho_k = \{\tilde{B}_k, T\tilde{B}_k, \dots, T^{u_k q_{2n_k}-1}\tilde{B}_k\}$$

of Rokhlin towers for T with heights $u_k q_{2n_k}$, $k \geq 1$. Since

$$\mu\Big(\bigcup_{j=0}^{u_k q_{2n_k}-1} T^j \tilde{B}_k\Big) = 1 - \tau_k, \quad \tau_k \longrightarrow 0 \tag{37}$$

and the diameter of each level of ϱ_k is smaller than $|I_k|$ ($|I_k|$ goes to zero as k goes to ∞), the sequence $\{\varrho_k\}$ tends to the point partition when k goes to ∞.

From the definition of the $\mathbf{R}$-cocycle φ, it follows that the $\mathbf{S}^1$-cocycle $e^{2\pi i\varphi}$ takes constant values on each level of ϱ_k except for the set

$$E_k = \bigcup_{j=0}^{a_{2n_k+1}q_{2n_k}-1} T^j J_1^{k+1}.$$

Moreover

$$\mu(E_k) < a_{2n_k+1} q_{2n_k} l_{k+1} < \frac{1}{a_{2n_{k+1}+1}}. \tag{38}$$

Let $B_k = \{x \in \tilde{B}_k \setminus E_k\colon T^{iu_kq_{2n_k}}x \in \tilde{B}_k \setminus E_k,\ i = 1,\dots,\tilde{q}_k - 1\}$. Now, take

$$F_k = B_k \times [0, \frac{1}{\tilde{q}_k}).$$

Notice that $E_k \times [0,1)$ is disjoint from $\bigcup_{j=0}^{\tilde{q}_k u_k q_{2n_k}-1}(T_{e^{2\pi i\varphi}})^j F_k$ and that the set

$$\mathcal{R}_k = \{F_k, \dots, (T_{e^{2\pi i\varphi}})^{\tilde{q}_k u_k q_{2n_k}-1}F_k\}$$

is a Rokhlin tower for $T_{e^{2\pi i\varphi}}$. This latter fact follows from the observation that the product of the values of $e^{2\pi i\varphi}$ across the tower $\varrho_k \cap ([0,1) \setminus E_k)$ is constant and equal to $e^{2\pi i\frac{\tilde{p}_k}{\tilde{q}_k}}$ (we use here the definition of φ and Lemma 1). The levels of $\mathcal{R}_k$ have diameters smaller than

$$l_k\frac{a_{2n_k+1}}{u_k} + \frac{1}{\tilde{q}_k} < \frac{1}{u_kq_{2n_k}} + \frac{1}{\tilde{q}_k}$$

and the latter number goes to zero by (36). Hence, it remains to prove that

$$\mu \times \mu(\bigcup_{j=0}^{\tilde{q}_k u_k q_{2n_k}-1} (T_{e^{2\pi i\varphi}})^j F_k) \longrightarrow 1.$$

We have

$$\mu(\bigcup_{j=0}^{\tilde{q}_k u_k q_{2n_k}-1} (T_{e^{2\pi i\varphi}})^j F_k) = u_kq_{2n_k}\tilde{q}_k(\mu\times\mu)(F_k) = u_kq_{2n_k}\mu(B_k).$$

In view of (34),(35) and (37) we obtain that

$$\mu(B_k) \geq \mu(\tilde{B}_k) - \mu(E_k) - l_k\tilde{q}_k = \frac{1}{u_kq_{2n_k}} - \mathrm{o}\Big(\frac{1}{u_kq_{2n_k}}\Big)$$

and the result follows. ∎

Remark 2: In [1] Anosov and Katok construct examples of C^∞ transformations on any two-dimensional compact manifold. The class of examples they construct include ones which are rigid, rank-1 and weakly mixing. If metric examples can be constructed which are both weakly isomorphic but not isomorphic, and are sufficiently well cyclically approximated to admit the Anosov and Katok construction, then their method would give C^∞ examples. As their method is actually a construction on a disk, vanishing to all orders on the boundary of the disk, their method cannot possibly give analytic examples.

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